Eur. Phys. J. D **3**, 123–127 (1998)

THE EUROPEAN PHYSICAL JOURNAL D

Crystallized vortex crystals

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Received: 24 February 1998 / Revised: 4 March 1998 / Accepted: 4 May 1998

Abstract. Numerical simulations of the equations of motion of 300 charged particles confined to a plane with an additional magnetic field orthogonal to the plane reproduce recently observed self-organization of non-neutral plasmas into a small number of interacting vortices. In the presence of damping we observe crystallized vortices, *i.e.* vortices with regular internal structure. We also observe crystallized vortex crystals, *i.e.* geometric patterns of crystallized vortices. Fractal vortex arrangements are investigated and found to be stable. Our results are relevant for quantum dots and artificial atoms.

PACS. 36.40.-c Atomic and molecular clusters – 32.80.Pj Optical cooling of atoms; trapping – 52.25.Wz Nonneutral plasmas

The experimental discovery of vortex crystals [1] marks the birth of a new paradigm in the physics of charged plasmas [2]. The plasma vortices correspond to regions of enhanced plasma density. They are called vortices because two-dimensional one-component plasmas are well described by hydrodynamic equations and the plasma density is directly proportional to the vorticity of the fluid analog of the plasma [1–3]. The relationship between vorticity and plasma density is intuitively clear since because of the Lorentz force regions of higher plasma density rotate faster. Vortex crystals are created by self-organization of a quasi two-dimensional dense gas of strongly interacting charged particles stored in a cylindrical trap with axial magnetic field [1]. The observed geometric patterns of vortex crystals resemble those of ion crystals that have been seen before as ordered structures in electrodynamical traps [4–9]. The patterns are understood as minimal energy configurations of vortices, *i.e.*, the result of the vortices' tendency to minimize the energy of their mutual electrostatic repulsion. We believe that the results obtained in this paper are also of relevance for quantum dots in strong magnetic fields [10,11]. This is so because quantum dots with a controlled number of electrons can nowadays be fabricated [12]. Moreover, in quantum dots the electrons are naturally confined to two-dimensional sheets [10,11]. Thus in this case the two-dimensionality imposed by our computations is a very good approximation.

In this paper we report the results of our numerical simulations of two-dimensional plasmas of charged particles in a strong magnetic field. Starting from low-entropy particle configurations our computations reproduce the tendency of the charged plasma to organize itself into a small number of vortices. Adding a small damping force we observe the formation of internal order of the vortices, very similar to the ordered geometric patterns of cold electron crystals predicted to occur as the lowest energy configurations in storage rings [13–15]. Thus, we obtain crystallized vortices. Crystallized vortices may or may not form a geometric pattern. If they do, we obtain a crystallized vortex crystal. To prove the existence of crystallized vortices and crystallized vortex crystals is the central focus of this paper.

Crystallized vortex crystals show order on two vastly different length scales. The large length scale corresponds to the distances between vortices in their ordered crystal arrangement. This is the length scale observed in the experiments [1]. The small scale corresponds to the ordered lattice of electrons within the vortices themselves. A natural question to ask is the following: Is it possible to produce ordered behaviour on more than two lengths scales? This is indeed the case. We construct fractal vortex crystals with order on many length scales. Our numerical simulations show that such arrangements are stable over the observation time allowed by our computational resources.

The formation of self-organized vortices is not restricted to electron plasmas. Ion plasmas are expected to show the same organization phenomena. In the case of an ion plasma laser cooling [16,17] offers a viable method for implementing the damping necessary to produce crystallized vortices. The type of ion species is not critical. Crystallized vortex crystals should be observable with any type of ions. A convenient ion species, *e.g.*, are Mg^+ ions for which efficient laser cooling has already been demonstrated (see, *e.g.*, [5,8,9]).

We now turn to the details of our numerical simulations. In order to demonstrate the self-organization phenomenon we solve the Newtonian equations of motion of 300 ions restricted to move in the (x, y) plane. The ions

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interact *via* their mutual Coulomb repulsion in the presence of a strong magnetic field parallel to the z-direction. The two-dimensional model is chosen mainly for numerical convenience. It is, however, an excellent approximation to experimental situations such as, *e.g.*, quantum dots [10,11] and plasma columns in cylindrical Penning traps [1,2]. The equations of motion of our model are given by

$$m\ddot{\mathbf{r}}_i = q\dot{\mathbf{r}}_i \times \mathbf{B} - \gamma\dot{\mathbf{r}}_i + \frac{q^2}{4\pi\epsilon_0}\sum_{j\neq i}^{N}\frac{\mathbf{r}_i - \mathbf{r}_j}{\mid \mathbf{r}_i - \mathbf{r}_j \mid^3}, \qquad (1)$$

where m is the mass of the particles, $\mathbf{r}_i$ is the position of particle number i, q is the charge of the particles, $\mathbf{B}$ is the applied magnetic field, γ is the damping constant and ϵ_0 is the dielectric constant. With the help of the cyclotron frequency $\omega_c = qB/m$ we define the unit of time as $T = 1/\omega_c$. The unit of length is chosen to be $l = 1$ μm. Introducing the dimensionless time $\tau = t/T$, the dimensionless positions $\rho = r/l$, the dimensionless damping constant $\Gamma = \gamma/(qB)$ and the coupling constant $c = m/(4\pi\epsilon_0 B^2 l^3)$ we obtain the dimensionless equations of motion

$$\ddot{\boldsymbol{\rho}}_i = \dot{\boldsymbol{\rho}}_i \times \hat{\mathbf{z}} - \Gamma\dot{\boldsymbol{\rho}}_i + c\sum_{j\neq i}^{N}\frac{\boldsymbol{\rho}_i - \boldsymbol{\rho}_j}{\mid \boldsymbol{\rho}_i - \boldsymbol{\rho}_j \mid^3}, \qquad (2)$$

where $\hat{\mathbf{z}}$ is the unit vector in the z-direction. We choose $c = 0.1$ for all the numerical computations reported below. For the demonstration of the self-organization phenomenon we choose a low-entropy initial condition of the ion plasma. The initial velocities of the 300 ions are chosen to be zero. The positions of the ions are chosen randomly within a two-dimensional ring of diameter $d = 100$ and width $w = 1$ (see Fig. 1a). Initial conditions of this type are generated experimentally by using appropriately shaped electrodes [1] or ion sources. Once the initial conditions are chosen at $\tau = 0$, we solve the equations of motion (2) for $\Gamma = 0$ over a long period of time $0 < \tau < \tau_{max} = 40\,000$. This time interval corresponds to more than 6000 cyclotron revolutions. Because of the relatively low plasma density in our computations radiative damping is negligible in this time interval. Freeze frames of the resulting motion after $\tau = 3000, 16\,000$ and $\tau = 39\,990$ are shown in Figures 1b–d, respectively. Figure 1b shows the initial break-up of the ring-configuration of particles into a large number of small vortices. In the course of time, small vortices merge to form larger vortices. This is shown in Figure 1c. Figure 1d, finally, shows that a small number of large vortices is stable for a very long time. The organization of the initially random ring of particles into rotating vortices separated by gaps of low ion density is clearly visible.

For $\Gamma = 0$ our simulations did not produce geometrically ordered highly symmetric vortex crystals as found in the experiments [1]. This may have two reasons. (i) Vortex crystals are obtained only in the presence of a damping force or (ii) we did not wait long enough for the vortices to settle down into a geometric pattern.

The following argument is in favour of (i). There is no *a priori* reason why vortices should always settle down into a crystalline configuration. Suppose we start with $N \geq 3$ vortices consisting of n_k, $k = 1, ..., N$ particles each. If the vortices are well separated from each other, we may approximate the complicated $\sum n_k$-body problem by an N-body problem of vortices with charge $n_k q$ each. But even the resulting effective problem of N bodies is still a complicated many-body problem which is known to possess chaotic solutions [18]. Thus in the absence of damping and for $N \geq 3$ it is not clear why the ordered state should be favoured over the chaotic state. Only in the case $N = 2$ is there always an ordered "crystal" since the two-body problem separates and thus is exactly integrable and therefore non-chaotic. The case $N = 2$ corresponds to a "double star" configuration which, we checked, is stable over as long a period of time as we were able to follow it numerically ($\tau_{max} = 4.5 \times 10^5$ for $n_1 = n_2 = 50$ particles).

The following observation is in favour of argument (ii). As is evident, *e.g.*, from Figure 1d the regions between the prominent vortices are filled with a low-density "background gas" consisting of particles that are not (yet) absorbed into the vortices. The vortices have to move through the background gas which, although the system is Hamiltonian and the total energy is conserved, may result in the damping of some macroscopic degrees of freedom such as the center-of-mass coordinates of the vortices. Thus, by this mechanism, and given enough time, we can imagine the vortices to crystallize. We were not able to follow our 300-particle ensemble long enough to see any significant damping by the background gas. Nevertheless this mechanism may be responsible for the formation of vortices in actual experiments [1].

In order to prove the existence of crystallized vortices we switched on the damping in our numerical simulations. This was done in the following way. We started an ensemble of 300 particles as depicted in Figure 1a. We then evolved this ensemble without damping ($\Gamma = 0$) for a time interval of $0 < \tau < 20\,000$. At $\tau = 20\,000$ we switched on the damping force with $\Gamma = 0.1$. Figure 2 shows the resulting particle configuration at $\tau = 21\,000$. It is clearly visible that the five major vortices have acquired a regular internal structure. We recommend this two-step procedure for creating crystallized vortices experimentally. In practice this two-step process may be realized by first evolving the particle ensemble over a time interval Δt_0 without laser cooling, switching on laser cooling only after the waiting time Δt_0. The ordered internal structure of the crystallized vortices reminds of the structure of a Wigner crystal [19].

We now turn to the central point of this paper, the demonstration of the existence of crystallized vortex crystals. In order to contrast the crystallized vortex crystal with the established phenomenon of vortex crystals [1] we examine once more the results of the computer simulation of the 300-ion ensemble illustrated in Figure 1. This time we pick a freeze frame at $\tau = 20\,000$. It is shown in Figure 3a. We see a near-geometric arrangement of vortices not unlike those seen in the experiments [1]. Examining their internal structure Figure 3a suggests that the locations of the ions within a vortex are close to random.

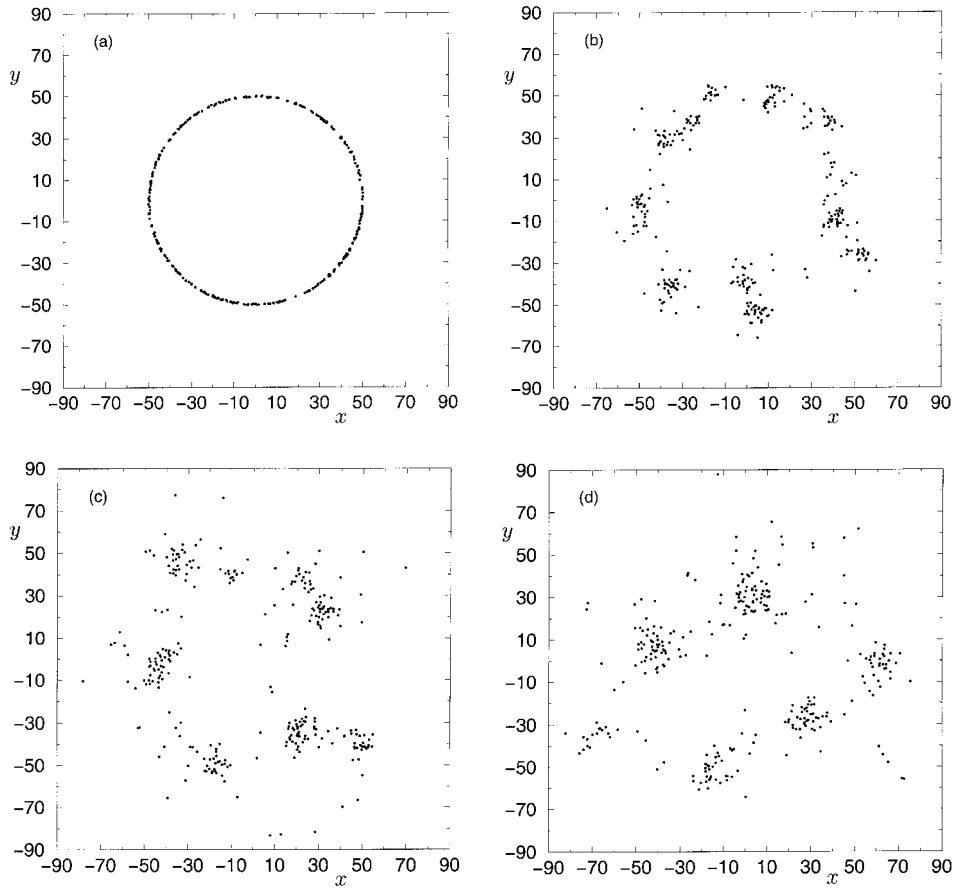

Fig. 1. Freeze frames of the time evolution of a cloud of 300 particles started on a ring of diameter $d = 100$ and width $w = 1$ with random positions and initial velocity zero. No damping is present in the computations ($\Gamma = 0$). (a) Initial condition ($\tau = 0$), (b) $\tau = 3000$, (c) $\tau = 16\,000$, (d) $\tau = 39\,990$.

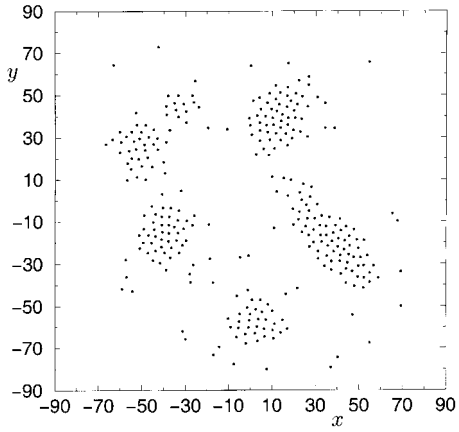

Fig. 2. Formation of ordered internal structure of a collection of vortices in the presence of damping.

This is consistent with the dynamical picture developed above, namely that the dynamics of strongly interacting charged particles in a magnetic field is chaotic. In order to substantiate the irregular nature of the ion locations we present in Figure 3b the probability distribution of the nearest neighbour spacings. The x-axis of this diagram is the nearest neighbour spacing normalized to the average spacing of ions in Figure 3a. The normalized spacing is denoted by s. The histogram in Figure 3b shows the spacing probability taken from the 300 data points of Figure 3a. The smooth line in Figure 3b is the spacing probability expected on the basis of ions sprinkled at random and with uniform distribution onto the (x, y) plane [20]. Interestingly this distribution is a "Wigner distribution" which also occurs in random matrix theory [20,21]. The numerical data (histogram) in Figure 3b are consistent with the Wigner distribution (full line). This lends strong support to the conjecture that there is no internal order in the ion arrangement shown in Figure 3a. Further support is derived from a computation of the plasma coupling parameter Γ_c. It is the ratio of the average Coulomb interaction energy between two ions and their average kinetic energy [13–15]. Molecular dynamics computations established [13–15] that there are three important regimes for the coupling parameter. (i) $\Gamma_c \ll 1$, (ii) $\Gamma_c \sim 1$ and (iii) $\Gamma_c \gg 1$. The particles behave like a gas in regime (i), they exhibit liquid behaviour in regime (ii) and they crystallize in regime (iii). Computing Γ_c for the ion configuration shown in Figure 3a we obtain $\Gamma_c \sim 0.1 \ll 1$. This provides independent confirmation that the internal structure of uncooled vortices is gas-like and not ordered.

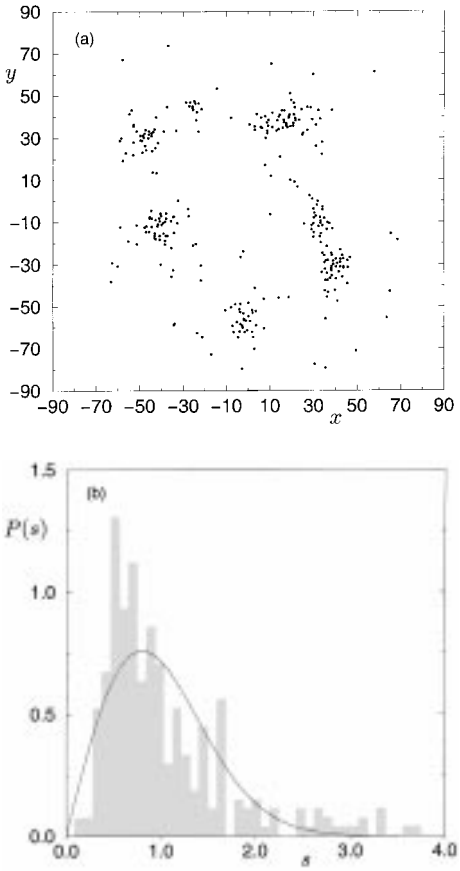

Fig. 3. Near-geometric arrangement of large vortices without internal order. (a) Freeze frame ($\tau = 20\,000$) of 300 ions started from the same initial condition as in Figure 1a. No damping is present ($\Gamma = 0$). The histogram in (b) shows the normalized nearest neighbour spacing distribution of the particles whose positions are shown in (a). The smooth line is the nearest neighbour spacing distribution corresponding to uncorrelated particles sprinkled randomly and uniformly on the (x, y) plane.

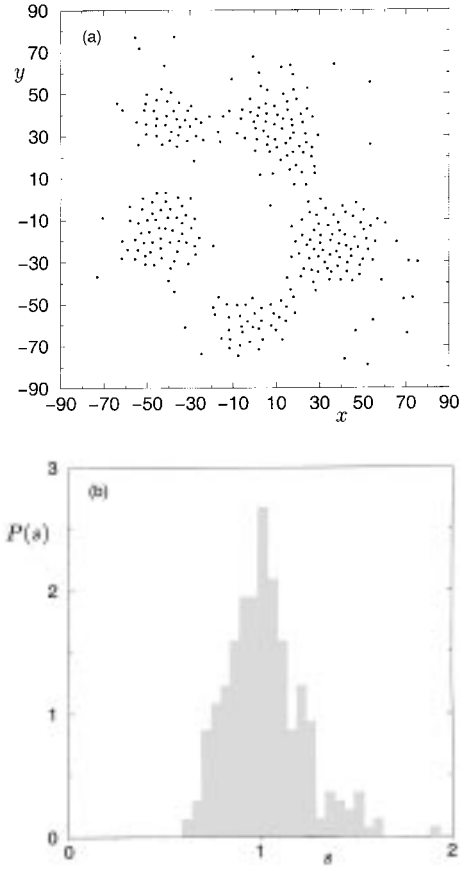

Fig. 4. Crystallized vortex crystal obtained from the same initial condition as shown in Figure 1a, but with the damping switched on ($\Gamma = 0.01$). (a) In contrast to Figure 3a the vortices show internal crystalline structure and the vortices themselves are ordered geometrically. (b) The nearest neighbour distribution. It is clearly peaked at the lattice constant ($s = 1$) and shows a void at small distances.

We now repeat the run illustrated in Figure 1, but with damping ($\Gamma = 0.01$) switched on right from the start. Examining the ion configuration at $\tau = 20\,000$, we obtain the ion locations shown in Figure 4a. This time the vortices arrange in a regular geometric pattern reminiscent of both, the ion crystals obtained in Paul traps [5,6,8, 9] and the vortex crystals obtained in experiments with one-component plasmas [1]. But in addition to the order on the largest length scale (the scale of the five vortices) we also observe that the internal structure of the vortices is now far from random. The internal structure itself resembles a Coulomb crystal as obtained, *e.g.*, in molecular dynamics computations for crystallized beams in ion storage rings [14,15]. Figure 4a shows that the internal crystalline structure of the vortices is not perfect. It contains faults and dislocations. The reason is probably that we had to choose a relatively large damping in order to make the computations feasible. This however leads to a relatively fast cooling process resulting in an imperfect crystal. Nevertheless it is clear from Figure 4a that the sub-structure of the five vortices is dominated by strong correlations and order. Thus, the vortex crystal shown in Figure 4a is qualitatively different from the vortex crystal shown in Figure 3a. In order to distinguish the two types of crystal from each other we coined the name "crystallized vortex crystal" which concisely describes the novel features of the vortex crystal shown in Figure 4a. In order to support the claim of an ordered sub-structure of the vortex crystal shown in Figure 4a we computed the nearest neighbour spacing statistics of the ion arrangement in Figure 4a. It is shown as the histogram in Figure 4b, again *versus* the normalized spacing s. Comparing the probability distribution in Figure 4b with the probability distribution in Figure 3b we see that the two distributions are qualitatively different. While the distribution in Figure 3b is consistent with a random ensemble of ions, the distribution in Figure 4b is peaked at the lattice constant ($s = 1$) and is vanishingly small for a large neighbourhood around $s = 0$. Both the peak at $s = 1$ and the void in the interval $0 < s < 0.5$ support the notion of a crystalline structure. We also computed the plasma coupling parameter Γ_c for

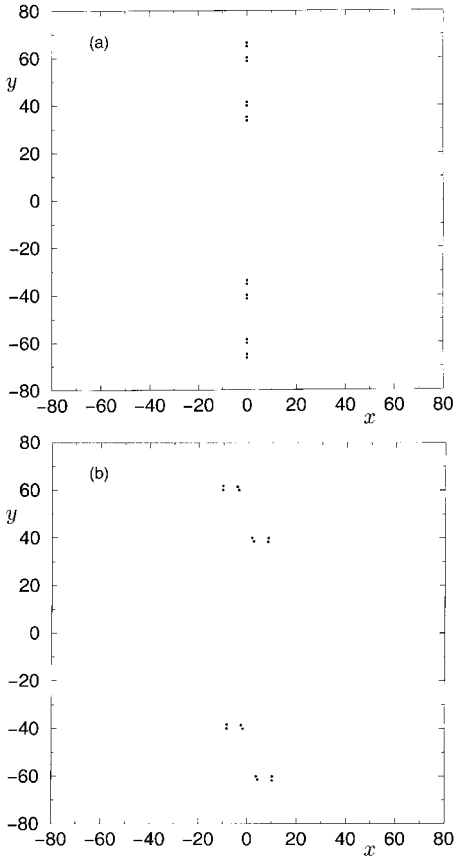

Fig. 5. Self-similar fractal vortex crystal.

the crystallized vortex crystal and found it to be far in excess of 100. This is easily explained. Over the time interval $\Delta\tau = 20\,000$ used to produce Figure 4a all the random motion of the ions is essentially dissipated ($\Gamma\Delta\tau = 200$). Thus we are deep in the strong coupling regime (iii) where crystallization is expected. Thus Figure 4 establishes the existence of crystallized vortex crystals.

Our fourth and final numerical simulation concerns the creation of a fractal vortex crystal. A fractal vortex crystal is the natural continuation of the idea of having order on two length scales (such as in the case of a crystallized vortex crystal) to a vortex crystal with order on many length scales. Thus a fractal vortex crystal consists of particles orbiting each other orderly on many different length scales. The simplest fractal vortex crystal is a self-similar vortex crystal. We illustrate the idea of a self-similar vortex crystal with a set of particles arranged in hierarchies of two. It is shown in Figure 5. The initial condition consists of 16 particles arranged symmetrically on the y axis as shown in Figure 5a. It consists of four hierarchies of dumbells with a self-similarity factor of 5. We followed the motion of this arrangement over the time interval $0 < \tau < 10\,000$. Figure 5b shows the final result of the particle positions at $\tau = 10\,000$. It proves that the crystal is stable over the observation time.

In this paper we presented the results of two-dimensional numerical simulations of up to 300 strongly coupled particles under the combined influence of their mutual Coulomb interaction and an additionally applied strong magnetic field orthogonal to the plane of motion of the particles. We were able to show the organization of particles into regions of large particle density, called vortices, and a practically empty space between them filled only by a low-density background gas formed by particles that are not incorporated into the high-density vortices. In the absence of damping we were not able to see the arrangement of vortices into regular geometric patterns. In the presence of damping we obtained crystallized vortices as well as geometric patterns of crystallized vortices, *i.e.* crystallized vortex crystals. In addition we showed that fractal vortex crystals exist and are stable. Fractal vortex crystals are particle ensembles organized in clusters that orbit each other on ever finer length scales. We hope that crystallized vortex crystals as well as fractal vortex crystals may soon be investigated experimentally. This is possible since some species of particles allow for efficient laser cooling which serves as an implementation of the damping mechanism present in the equations of motion (1).

R.B. is grateful for financial support by the Deutsche Forschungsgemeinschaft (SFB 276).

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